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Pseudopotentials, Lax equations and Bäcklund transformations for non-linear evolution equations

Maria Clara Nucci†

School of Mathematics, Georgia Institute of Technology, Atlanta, GA 30332, USA

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Abstract. It is shown how to generate both Lax equations and Bäcklund transformations for non-linear evolution equations using the concept of pseudopotentials associated with the properties of the Riccati equation. Several examples are given: modified Korteweg-de Vries, Harry Dym, Kaup-Kupershmidt and Sawada-Kotera equations.

1. Introduction

Innumerable papers are dedicated to the study of non-linear evolution equations and to the various techniques for solving them. Kaup (1980a) pointed out the importance of the pseudopotentials introduced by Wahlquist and Estabrook (1975). Hermann (1976) interpreted the prolongation structure as a connection; his equation (5) can be viewed as an assumption of Riccati equations for the pseudopotential of the Korteweg-de Vries equation. Crampin (1978) deduced the Lax equations and the Bäcklund transformations from the fact that the SL(2, R) connection associated with the soliton equation has zero curvature. Omote (1986) applied the correlated prolongation structure technique to the sine-Gordon equation, the Ernst equation and the chiral model; he obtained, generating both finite-dimensional and infinite-dimensional Lie algebras for the commutators, Bäcklund transformations and Lax equations respectively. In this paper, it will be shown by several examples that, using the direct approach (Corones and Testa 1976) one can obtain both the Lax equations and the Bäcklund transformations by requiring that the equations satisfied by the pseudopotentials be of a Riccati type. By means of the usual linearising transformation, one then obtains the Lax equations. Alternatively, by imposing that the Möbius group be an invariant transformation for the pseudopotential, one can easily obtain a Bäcklund transformation for the original non-linear evolution equation.

2. Modified Korteweg–de Vries equation

We consider the following form of the modified Korteweg-de Vries equation (Konopelchenko and Dubrovsky 1984):

$$q_t = q_{xxx} - 6q^2 q_x. \tag{2.1}$$

† Permanent address: Dipartimento di Matematica, Università di Perugia, 06100 Perugia, Italy.

We assume there exists a pseudopotential $u = u(x, t)$ such that

$$u_x = F(u, q) \tag{2.2a}$$

$$u_t = G(u, q, q_x, q_{xx}). \tag{2.2b}$$

Then requiring $u_{xt} = u_{tx}$ when (2.1) is satisfied, gives

$$F_u G + F_q(q_{xxx} - 6q^2 q_x) = G_u F + G_q q_x + G_{q_x} q_{xx} + G_{q_{xx}} q_{xxx} \tag{2.3}$$

from which we obtain

$$G = F_q q_{xx} + C(u, q, q_x) \tag{2.4}$$

$$C = -\tfrac{1}{2} F_{qq} q_x^2 + (F_u F_q - F F_{qu}) q_x + D(u, q) \tag{2.5}$$

$$F = \tfrac{1}{2} x_1(u) q^2 + x_2(u) q + x_3(u) \tag{2.6}$$

along with

$$[x_1, x_2] = [x_1, x_3] = 0 \tag{2.7}$$

where x_i $(i = 1, 2, 3)$ are functions of u only.

Here, and throughout this paper

$$[X, Y] \equiv X_u Y - X Y_u \tag{2.8}$$

and moreover:

$$[F, [F, F_q]] = 6F_q q^2 + D_q \tag{2.9}$$

$$[F, D] = 0. \tag{2.10}$$

From (2.7), we have the following two choices.

Case (a): $x_1 \neq 0$, $x_2 = A_2 x_1$, $x_3 = A_3 x_1$, $A_j = \text{constant}$ $(j = 2, 3)$

Case (b): $x_1 = 0$.

Since case (a) leads to trivial pseudopotentials, we have from (2.6), (2.9) and (2.10) that

$$F = x_2 q + x_3 \tag{2.11}$$

$$D = -2x_2 q^3 - \tfrac{1}{2}[x_2, [x_2, x_3]] q^2 - [x_3, [x_2, x_3]] q - x_4(u) \tag{2.12}$$

along with

$$[x_3, x_4] = 0 \qquad \text{i.e. } x_4 = c_4 x_3 (c_4 = \text{constant}) \tag{2.13}$$

$$2(-2 + \tfrac{1}{2} x_2'^2 - x_2 x_2'')[x_2, x_3] + x_2^2 (x_2''' x_3 - x_2 x_3''') = 0 \tag{2.14}$$

$$(x_2' x_3' - x_2 x_3'' - x_3 x_2'')[x_2, x_3] + x_2 x_3 (x_2''' x_3 - x_2 x_3''') = 0 \tag{2.15}$$

$$(c_4 + x_3'^2 - 2x_3 x_3'')[x_2, x_3] + x_3^2 (x_2''' x_3 - x_2 x_3''') = 0. \tag{2.16}$$

Here, and throughout this paper,

$$x_i'(u) \equiv \mathrm{d}x_i/\mathrm{d}u \qquad i = 1, 2, 3, 4. \tag{2.17}$$

Now (2.2) will be of a Riccati type if

$$x_2'''(u) = x_3'''(u) = 0. \tag{2.18}$$

Finally, from (2.14)-(2.16) we obtain

$$x_2(u) = -\frac{4}{c_1} + \frac{(c_1 u + c_2)^2}{4c_1} \tag{2.19}$$

$$x_3(u) = \frac{c_4}{b_1} + \frac{(b_1 u + b_2)^2}{4b_1} \tag{2.20}$$

$$c_4 = -\frac{(c_1 b_2 - b_1 c_2)^2}{4c_1^2} + \frac{4b_1^2}{c_1^2} \tag{2.21}$$

where c_i and b_i $(i = 1, 2)$ are arbitrary constants.

Among the choices that we have, assume $b_2 = c_2 = 0$, $b_1 = 4\lambda$, $c_1 = -4\lambda$. Then (2.2) becomes

$$u_x = [(1 - \lambda^2 u^2)q + 1 + \lambda^2 u^2]/\lambda \tag{2.22a}$$

$$u_t = [(1 - \lambda^2 u^2)q_{xx} + 4\lambda u q_x - 2(1 - \lambda^2 u^2)q^3 \\ - 2(1 + \lambda^2 u^2)q^2 - 4(1 - \lambda^2 u^2)q - 4(1 + \lambda^2 u^2)]/\lambda \tag{2.22b}$$

where λ represents the spectral parameter.

Now if we introduce the linearising transformation

$$u = v_1/v_2 \tag{2.23}$$

we obtain the following Lax equations in the AKNS form (Ablowitz and Segur 1981):

$$V_x = AV \tag{2.24a}$$

$$V_t = BV \tag{2.24b}$$

where

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.25}$$

$$A = \begin{pmatrix} 0 & q+1 \\ \lambda^2(q-1) & 0 \end{pmatrix} \frac{1}{\lambda} \tag{2.26}$$

$$B = \begin{pmatrix} 2\lambda q_x & -(-q_{xx} + 2q^3 + 2q^2 + 4q + 4) \\ \lambda^2(q_{xx} - 2q^3 + 2q^2 - 4q + 4) & -2\lambda q_x \end{pmatrix} \frac{1}{\lambda}. \tag{2.27}$$

Alternatively, another solution q^* of (2.1) will have a pseudopotential u^* such that from (2.22a)

$$u_x^* = [(1 - \lambda^{*2} u^{*2})q^* + 1 + \lambda^{*2} u^{*2}]/\lambda^*. \tag{2.28}$$

If we assume

$$u^* = -u \qquad \text{and} \qquad \lambda^* = \lambda \tag{2.29}$$

then, substituting in (2.28), we obtain

$$u_x = [-(1 - \lambda^2 u^2)q^* - (1 + \lambda^2 u^2)]/\lambda. \tag{2.30}$$

Combining (2.22a) and (2.30) gives

$$q + q^* = -2\cosh\left(\int (q - q^*)\,\mathrm{d}x\right) \tag{2.31}$$

the spatial part of the well known Bäcklund transformation for (2.1). Note that from (2.22a) the relation between the pseudopotential u and the eigenfunction ψ of the operator L in Konopelchenko and Dubrovsky (1984), if $\lambda = 1$, is found to be

$$u = \frac{\psi - \psi_x}{\psi + \psi_x} \tag{2.32}$$

with the corresponding eigenvalue being equal to -1.

3. Harry Dym equation

We consider the following form of the Harry Dym equation (Calogero and Degasperis 1982):

$$q_t = q^3 q_{xxx}. \tag{3.1}$$

The equations satisfied by the pseudopotential u are in the form (2.2). By the same method developed in the previous section, we obtain

$$F = -\tfrac{1}{2}x_2(u)q^{-2} - x_3(u) \tag{3.2}$$

$$G = F_q q^3 q_{xx} + C(u, q, q_x) \tag{3.3}$$

$$C = (F_u F_q - F F_{qu}) q^3 q_x + D(u,q) \tag{3.4}$$

$$D = \tfrac{1}{2}[x_2, [x_2, x_3]]q^{-1} - [x_3, [x_2, x_3]]q + x_4(u) \tag{3.5}$$

along with

$$[x_2, x_4] = [x_3, x_4] = 0 \tag{3.6}$$

$$x_2(u) = (c_1 u + c_2)^2 \tag{3.7}$$

$$x_3(u) = (b_1 u + b_2)^2 \tag{3.8}$$

where b_i and c_i ($i = 1, 2$) are arbitrary constants. If we make the choice $c_1 = -\lambda$, $c_2 = 1$, $b_1 = \lambda/\sqrt{2}$ and $b_2 = 1/\sqrt{2}$, then (2.2) becomes

$$u_x = -\tfrac{1}{2}(1 - \lambda u)^2 q^{-2} - \tfrac{1}{2}(1 + \lambda u)^2 \tag{3.9a}$$

$$u_t = (1 - \lambda u)^2 q_{xx} - 2\lambda(1 - \lambda^2 u^2) q_x + 2\lambda^2 (1 - \lambda u)^2 q^{-1} + 2\lambda^2 (1 + \lambda u)^2 q \tag{3.9b}$$

where λ is the spectral parameter.

By means of the transformation (2.23), we obtain the Lax equations (2.24) with:

$$A = \tfrac{1}{2}\begin{pmatrix} -\lambda(1 - q^{-2}) & -(1 + q^{-2}) \\ \lambda^2(1 + q^{-2}) & \lambda(1 - q^{-2}) \end{pmatrix} \tag{3.10}$$

$$B = \begin{pmatrix} -\lambda(q_{xx} + 2\lambda^2 q^{-1} - 2\lambda^2 q) & q_{xx} + 2\lambda q_x + 2\lambda^2 q^{-1} + 2\lambda^2 q \\ -\lambda^2(q_{xx} + 2\lambda q_x + 2\lambda^2 q^{-1} + 2\lambda^2 q) & \lambda(q_{xx} + 2\lambda^2 q^{-1} - 2\lambda^2 q) \end{pmatrix}. \tag{3.11}$$

Alternatively, from (3.9a) if we look for a solution q^* of (3.1) corresponding to $u^* = -u$ and $\lambda^* = -\lambda$, then we find

$$(q^*)^{-2} + q^{-2} = -2\left(1 - \tfrac{1}{2}\lambda \int [(q^*)^{-2} - q^{-2}]\,\mathrm{d}x\right)^2 \tag{3.12}$$

the spatial part of a Bäcklund transformation for (3.1). Note that the relation between the pseudopotential u and the eigenfunction ψ of the operator L (Konopelchenko and Dubrovsky 1984) is found, if $\lambda = 1$, to be

$$u = -\frac{\psi - \psi_x}{\psi + \psi_x} \tag{3.13}$$

with the corresponding eigenvalue being equal to -1.

4. Kaup–Kupershmidt equation

We consider the following equation (Kaup 1980b):

$$q_t = q_{xxxxx} + 5qq_{xxx} + \tfrac{25}{2}q_x q_{xx} + 5q^2 q_x. \tag{4.1}$$

The pseudopotential u will satisfy:

$$u_x = F(u, q) \tag{4.2a}$$

$$u_t = G(u, q, q_x, q_{xx}, q_{xxx}, q_{xxxx}). \tag{4.2b}$$

By the usual method, we obtain

$$F = (c_1 u + c_2)^2 q + 1/4c_1^2 \tag{4.3}$$

$$G = (c_1 u + c_2)^2 q_{xxxx} - \left(\frac{c_1 u + c_2}{2c_1}\right) q_{xxx} + \left(\frac{9(c_1 u + c_2)^2}{2} q + \frac{1}{8c_1^2}\right) q_{xx} + 4(c_1 u + c_2)^2 q_x^2$$
$$-2\left(\frac{c_1 u + c_2}{c_1}\right) qq_x + (c_1 u + c_2)^2 q^3 + \frac{q^2}{4c_1^2} \tag{4.4}$$

where c_i $(i = 1, 2)$ are arbitrary constants. If we choose $c_1 = \lambda$ and $c_2 = 0$, (4.2) becomes

$$u_x = \lambda^2 u^2 q + 1/4\lambda^2 \tag{4.5a}$$

$$u_t = \lambda^2 u^2 q_{xxxx} - \tfrac{1}{2}uq_{xxx} + (\tfrac{9}{2}\lambda^2 u^2 q + 1/8\lambda^2)q_{xx} + 4\lambda^2 u^2 q_x^2 - 2uqq_x + \lambda^2 u^2 q^3 + q^2/4\lambda^2 \tag{4.5b}$$

where λ is the spectral parameter. Using (2.23), the Lax equations are given by (2.24) with

$$A = \begin{pmatrix} 0 & 1/4\lambda^2 \\ -\lambda^2 q & 0 \end{pmatrix} \tag{4.6}$$

$$B = \begin{pmatrix} -(q_{xxx}/4 + qq_x) & (q_{xx}/2 + q^2)/4\lambda^2 \\ -\lambda^2(q_{xxxx} + 9qq_{xx}/2 + 4q_x^2 + q^3) & q_{xxx}/4 + qq_x \end{pmatrix}. \tag{4.7}$$

Alternatively, from (4.5a), choosing q^* to be another solution of (4.1) such that $u^* = -u$ and $\lambda^* = -\lambda$, we obtain the following spatial part of a Bäcklund transformation for (4.1):

$$q + q^* = -\tfrac{1}{8}\left(\int (q - q^*)\, \mathrm{d}x\right)^2. \tag{4.8}$$

5. Sawada–Kotera equation

We consider the following equation (Sawada and Kotera 1974):

$$q_t = q_{xxxxx} + 5qq_{xxx} + 5q_xq_{xx} + 5q^2q_x. \tag{5.1}$$

The pseudopotential u will satisfy (4.2) being:

$$F = (c_1u + c_2)^2q + 1/c_1^2 \tag{5.2}$$

$$G = (c_1u+c_2)^2q_{xxxx} - (2/c_1)(c_1u+c_2)q_{xxx} + [3(c_1u+c_2)^2q + 2/c_1]q_{xx} + (c_1u+c_2)^2q_x^2 - (2/c_1)(c_1u+c_2)qq_x + (c_1u+c_2)^2q^3 + q^2/c_1^2 \tag{5.3}$$

with c_i ($i = 1, 2$) arbitrary constants. Assuming $c_1 = \lambda$ and $c_2 = 0$, (4.2) becomes

$$u_x = \lambda^2u^2q + 1/\lambda^2 \tag{5.4a}$$

$$u_t = \lambda^2u^2q_{xxxx} - 2uq_{xxx} + (3\lambda^2u^2q + 2/\lambda^2)q_{xx} + \lambda^2u^2q_x^2 - 2uqq_x + \lambda^2u^2q^3 + q^2/\lambda^2. \tag{5.4b}$$

From (5.4), the Lax equations are given by (2.24) with

$$A = \begin{pmatrix} 0 & 1/\lambda^2 \\ -\lambda^2q & 0 \end{pmatrix} \tag{5.5}$$

$$B = \begin{pmatrix} -(q_{xxx} + qq_x) & (2q_{xx} + q^2)/\lambda^2 \\ -\lambda^2(q_{xxxx} + 3qq_{xx} + q_x^2 + q^3) & q_{xxx} + qq_x \end{pmatrix}. \tag{5.6}$$

Alternatively, by means of (2.29), we obtain from (5.4a)

$$q^* + q = -\tfrac{1}{2}\left(\int (q^* - q)\,\mathrm{d}x\right)^2 \tag{5.7}$$

the spatial part of a Bäcklund transformation for (5.1).

6. Final comments

The results of §§ 2–5 have displayed a procedure to obtain Lax equations and Bäcklund transformations for non-linear evolution equations using the concept of pseudopotentials associated with the properties of the Riccati equations. Other Lax equations and Bäcklund transformations could be exhibited by means of different choices of the arbitrary constants. It is left as an exercise for the diligent reader to find the time part of the Bäcklund transformations (2.34), (3.13), (4.8) and (5.7). Note that the transformation (2.29) could be considered a gauge-like invariance transformation.

References

Ablowitz M J and Segur H 1981 *Solitons and the Inverse Scattering Transform* (Philadelphia: SIAM)

Calogero F and Degasperis A 1982 *Spectral Transform and Solitons* (Amsterdam: North-Holland)

Corones J and Testa F J 1976 *Pseudopotentials and their Applications* (*Lecture Notes in Mathematics* **515**) (Berlin: Springer) p 184

Crampin M 1978 *Phys. Lett.* **66A** 170

Hermann R 1976 *Phys. Rev. Lett.* **36** 835
Kaup D J 1980a *Physica* **1D** 391
—— 1980b *Stud. Appl. Math.* **62** 189
Konopelchenko B G and Dubrovsky V G 1984 *Phys. Lett.* **102A** 15
Omote M 1986 *J. Math. Phys.* **27** 2853
Sawada K and Kotera J 1974 *Prog. Theor. Phys.* **51** 1355
Wahlquist H D and Estabrook F B 1975 *J. Math. Phys.* **16** 1